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LETTER TO THE EDITOR

An exact solution to two-dimensional Korteweg–de Vries–Burgers equation

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Abstract. By applying a special solution of square Hopf–Cole type to an ordinary differential equation, we propose a bounded travelling wave solution $u(x, y, t) = v(\xi) = v(kx + ly - \omega t)$ to the two-dimensional Korteweg–de Vries–Burgers equation is monotonic and possesses an inflection point with respect to ξ.

Integrable systems, both classical and quantum, are a fascinating subject. Decades of research in this area have led to mathematical developments which are quite beautiful. However, not all systems posed in physics are integrable (see Kruskal *et al* [1]), for instance, the Korteweg–de Vries–Burgers (KdV–Burgers) equation. Therefore the direct methods to solve nonlinear systems appear to be more powerful and important. In this letter we will propose an exact solution to a general two-dimensional Korteweg–de Vries–Burgers (2DKdV–Burgers for short) equation

$$(u_t + 2auu_x + bu_{xx} + cu_{xxx})_x + du_{yy} = 0 \tag{1}$$

where a, b, c, d are constants, directly from the equation itself. Equation (1) is a two-dimensional generalization of KdV–Burgers equation served as a nonlinear wave model of fluid in an elastic tube (Johnson [2]), liquid with small bubbles (van Wijngaarden [3]) and turbulence (Gao [4]). Here we would like to construct an analytic solution of it by analysing an ordinary differential equation.

First we take the form of the required solution as follows

$$u(x, y, t) = v(\xi) \qquad \xi = kx + ly - \omega t$$

where k, l, ω are constants to be determined, and thus 2DKdV–Burgers equation (1) becomes

$$-\omega k v_{\xi\xi} + 2ak^2(vv_\xi)_\xi + bk^3 v_{\xi\xi\xi} + ck^4 v_{\xi\xi\xi\xi} + dl^2 v_{\xi\xi} = 0.$$

Integrating the above equation twice with regard to ξ, we obtain

$$-\omega k v + ak^2 v^2 + bk^3 v_\xi + ck^4 v_{\xi\xi} + dl^2 v = K \tag{2}$$

with the second integration constant K and the first one taken to be zero. Making the transformation

$$v = v_\pm = w_\pm + \frac{\omega k - dl^2 \pm \sqrt{(\omega k - dl^2)^2 + 4ak^2 K}}{2ak^2}$$

equation (2) may be written as

$$\pm\sqrt{(\omega k - dl^2)^2 + 4ak^2K}\,\omega_\pm + ak^2w_\pm^2 + bk^3w_{\pm\xi} + ck^4w_{\pm\xi\xi} = 0. \qquad (3\pm)$$

Next we search for solutions of the following ordinary differential equation

$$pw + qw^2 + rw_\xi + sw_{\xi\xi} = 0 \qquad (4)$$

with constants p, q, r, s. Let us introduce a square Hopf–Cole transformation $w = \alpha\phi_\xi^2/\phi^2$, $\alpha =$ constant. This moment we have

$$\begin{aligned} pw + qw^2 + rw_\xi + sw_{\xi\xi} &= \alpha p\frac{\phi_\xi^2}{\phi^2} + \alpha^2 q\frac{\phi_\xi^4}{\phi^4} + 2\alpha r\left(\frac{\phi_\xi\phi_{\xi\xi}}{\phi^2} - \frac{\phi_\xi^3}{\phi^3}\right) \\ &\quad + 2\alpha s\left(\frac{\phi_{\xi\xi}^2 + \phi_\xi\phi_{\xi\xi\xi\xi}}{\phi^2} - \frac{5\phi_\xi^2\phi_{\xi\xi}}{\phi^3} + \frac{3\phi_\xi^4}{\phi^4}\right) \\ &= \alpha[p\phi_\xi^2 + 2r\phi_\xi\phi_{\xi\xi} + 2s(\phi_{\xi\xi}^2 + \phi_\xi\phi_{\xi\xi\xi})]\phi^{-2} \\ &\quad - 2\alpha(r\phi_\xi^3 + 5s\phi_\xi^2\phi_{\xi\xi})\phi^{-3} + \alpha(\alpha q\phi_\xi^4 + 6s\phi_\xi^4)\phi^{-4}. \end{aligned}$$

Hence we may choose

$$\alpha q + 6s = 0 \qquad (5a)$$

$$r\phi_\xi + 5s\phi_{\xi\xi} = 0 \qquad (5b)$$

$$p\phi_\xi^2 + 2r\phi_\xi\phi_{\xi\xi} + 2s(\phi_{\xi\xi}^2 + \phi_\xi\phi_{\xi\xi\xi}) = 0. \qquad (5c)$$

By (5.1), $\alpha = -6s/q$, and by (5.2),

$$\phi = \phi(\xi) = F_1\,\mathrm{e}^{\lambda\xi} + F_2 \qquad \lambda = -\tau/5s$$

where F_1, F_2 are constants and we need a condition $F_1F_2 > 0$ to avoid $\phi = 0$. Now (5.3) reads as

$$(p + 2\lambda r + 4\lambda^2 s)\lambda^2F_1^2\,\mathrm{e}^{2\lambda\xi} = 0$$

which requires $p = 6r^2/25s$ in order to generate non-trivial solutions. Therefore when $p = 6r^2/25s$, equation (4) has a solution

$$w = -\frac{6s\lambda^2}{q}\frac{F_1^2\,\mathrm{e}^{2\lambda\xi}}{(F_1\,\mathrm{e}^{\lambda\xi} + F_2)^2} = -\frac{6r^2}{25qs}\frac{\exp[-(2r/5s)\xi]}{(\exp[-(r/5s)\xi] + E)^2} \qquad (6)$$

with an arbitrary constant $E > 0$. It is easy to calculate

$$w_\xi = \frac{rE\psi_\xi^2}{5q\psi^3} \qquad w_{\xi\xi} = 2\alpha E\lambda^4\,\mathrm{e}^{2\lambda\xi}(2E - \mathrm{e}^{\lambda\xi})\psi^{-4} \qquad (7)$$

where $\psi = \mathrm{e}^{\lambda\xi} + E$. Thus the solution (6) is monotonic and possesses an inflection point $\xi = (1/\lambda)\ln(2E)$. We note that under $w = \bar{w} - p/q$, (4) may be transformed into

$$-p\bar{w} + q\bar{w}^2 + r\bar{w}_\xi + s\bar{w}_{\xi\xi} = 0. \qquad (8)$$

The first and last coefficients of (8) are opposite sign since $ps = 6r^2/25$. Guan and Gao [5] have made some qualitative analyses for this kind of equation and consider it difficult to find solutions of (8). Here we have presented a special solution to (8) which belongs to the first type of Guan and Gao [5] because of (7).

In what follows, we want to find solutions of (3) by using (4) and (8). Naturally we need

$$25\,\mathrm{sgn}(c)c\sqrt{(\omega k-dl^2)^2+4ak^2K}=6b^2k^2 \tag{9}$$

which corresponds to $p=6r^2/25s$. Set

$$f(\xi)=-\frac{6b^2}{25ac}\frac{\exp[-(2b/5ck)\xi]}{(\exp[-(b/5ck)\xi]+E)^2} \qquad E>0$$

and obviously we have $|f(\xi)|\leqslant 6b^2/25|ac|$. If $c<0$, then we write equation (3+) as

$$-(-\sqrt{(\omega k-dl^2)^2+4ak^2K})w_+ + ak^2w_+^2+bk^3w_{+\xi}+ck^4w_{+\xi\xi}=0.$$

By means of (8), it has a solution

$$w_+=f(\xi)-\frac{\sqrt{(wk-dl^2)^2+4ak^2K}}{ak^2}$$

and by means of (4), equation (3−) has a solution $w_-=f(\xi)$. In this way, we see that

$$v=v_+=v_-=f(\xi)+\frac{\omega k-dl^2-\sqrt{(\omega k-dl^2)^2+4ak^2K}}{2ak^2}$$

solves (2). Similarly, when $c>0$, we can find a solution of (2)

$$v=v_+=v_-=f(\xi)+\frac{\omega k-dl^2+\sqrt{(\omega k-dl^2)^2+4ak^2K}}{2ak^2}.$$

Now summing up, we see that 2DKdV–Burgers equation (1) has a bounded exact solution with the following form

$$u(x,y,t)=v(\xi)=f(\xi)+\frac{\omega k-dl^2+\mathrm{sgn}(c)\sqrt{(\omega k+dl^2)^2+4ak^2K}}{2ak^2}$$

$$=-\frac{6b^2}{25ac}\frac{\exp[-(2b/5ck)(kx+ly-\omega t)]}{(\exp[-(b/5ck)(kx+ly-\omega t)]+E)^2}+\frac{\omega k-dl^2}{2ak^2}+\frac{3b^2}{25ac} \tag{10}$$

where $E>0$, k, l, ω, K satisfy (9). Our exact solution (10) contains four changeable constants. For example, we may arbitrarily decide

$$E>0 \qquad k\neq 0 \qquad l\in R \qquad \mathrm{sgn}(a)K\leqslant\frac{36b^4k^2}{2500|a|c^2}$$

but ω must be

$$\omega=\frac{dl^2}{k}\pm\sqrt{\frac{36b^4k^4}{625c^2}-4ak^2K}.$$

If we choose $K=0$, then

$$\omega=\frac{dl^2}{k}\pm\frac{6b^2k}{25\,\mathrm{sgn}(c)c}$$

and thus

$$\beta:=\frac{\omega k-dl^2}{2ak^2}+\frac{3b^2}{25ac}=(\pm\mathrm{sgn}(c)+1)\frac{3b^2}{25ac}.$$

When $\beta=0$ and $k=1$, $l=0$, the solution (10) has appeared in Jeffrey and Xu [6] and Halford and Vlieg-Hulstman [7]. When $\beta=6b^2/25ac$, we obtain another solution

$$u(x, y, t)=-\frac{6b^2}{25ac}\frac{\exp[-(2b/5ck)(kx+ly-\omega t)]}{(\exp[-(b/5ck)(kx+ly-\omega t)]+E)^2}+\frac{6b^2}{25ac}$$

with arbitrary constants $k(\neq 0)$, l, $E(>0)$ and $\omega=dl^2/k+6b^2k/25c$. Naturally, the function (10) with $d=0$ solves KdV–Burgers equation, i.e. equation (1) with $d=0$, which includes a particular analytic solution given by Xiong [8].

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